MATH 245 S25, Exam 2 Solutions

1. Carefully define the following terms: well-ordered by $<$, big O.

   We say that a set of numbers S is well-ordered by some order $<$ if every nonempty subset of S has a minimal element in the $<$ ordering. Given two sequences a_n, b_n, we say that a_n is big O of b_n if $\exists M \in \mathbb{R}, \exists n_0 \in \mathbb{N}, \forall n \geq n_0,\ |a_n| \leq M|b_n|$.

2. Carefully state the following theorems: Proof by Contradiction Theorem, Nonconstructive Existence Proof Theorem.

   The proof by contradiction theorem says that (for any propositions p, q) to prove $p \to q$, we can prove $p \wedge \neg q \equiv F$. The nonconstructive existence proof theorem says that (for any domain D and any predicate $P(x)$) to prove $\exists x \in D,\ P(x)$ we can prove $\forall x \in D,\ \neg P(x) \equiv F$.

3. Solve the recurrence that has initial conditions $a_0 = 0, a_1 = 2$ and relation $a_n = -2a_{n-1} + 2a_{n-2}$ $(n \geq 2)$.

   This recurrence has characteristic polynomial $r^2 + 2r - 2$, which has roots $r_1 = -1 + \sqrt{3}$ and $r_2 = -1 - \sqrt{3}$ (we can use the quadratic formula to find these). Since these are distinct, the general solution is $a_n = Ar_1^n + Br_2^n$. We now apply the initial conditions to get $0 = a_0 = Ar_1^0 + Br_2^0 = A + B$ and $2 = a_1 = Ar_1^1 + Br_2^1 = Ar_1 + Br_2$. Solving the system $\{0 = A + B, 2 = Ar_1 + Br_2\}$ we get $A = \frac{2}{r_1 - r_2} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $B = -A = \frac{-1}{\sqrt{3}}$. This gives specific solution $a_n = \frac{1}{\sqrt{3}}(-1+\sqrt{3})^n + \frac{-1}{\sqrt{3}}(-1-\sqrt{3})^n$, which may be simplified if desired as $a_n = \frac{(-1+\sqrt{3})^n - (-1-\sqrt{3})^n}{\sqrt{3}}$. CAUTION: $(-1+\sqrt{3})^n - (-1-\sqrt{3})^n$ cannot be simplified!

4. Let $x \in \mathbb{R}$. Use cases to prove that $\big||x+1| - |x-1|\big| \leq 2$.

   We split into three cases, based on x.
   Case $x \geq 1$: $|x+1| = x+1$ and $|x-1| = x-1$, so $\big||x+1|-|x-1|\big| = |(x+1)-(x-1)| = |2| \leq 2$.
   Case $x \leq -1$: Now $|x+1| = -(x+1)$ and $|x-1| = -(x-1)$, so $\big||x+1| - |x-1|\big| = |-(x+1) + (x-1)| = |-2| = 2 \leq 2$.
   Case $-1 \leq x \leq 1$: Now $|x+1| = x+1$ and $|x-1| = -(x-1)$, so $\big||x+1| - |x-1|\big| = |(x+1) + (x-1)| = |2x| = 2|x|$. Since $-1 \leq x \leq 1$ in this case, we have $|x| \leq 1$, so $2|x| \leq 2$.
   In all three cases, $\big||x+1| - |x-1|\big| \leq 2$.

5. Prove that for all $n \in \mathbb{N}_0$, we have $F_{n+1}(F_n + F_{n+2}) = \displaystyle\sum_{i=0}^{n} F_i^2 + F_{i+1}^2$. Here F_n denotes the Fibonacci numbers.

   Shifted induction (starting with 0). Base case $n = 0$: $F_n = 0, F_{n+1} = 1, F_{n+2} = 1$ so $F_{n+1}(F_n + F_{n+2}) = 1(0+1) = 1$, while $\sum_{i=0}^n F_i^2 + F_{i+1}^2 = F_n^2 + F_{n+1}^2 = 0^2 + 1^2 = 1$.
   Inductive case: Let $n \in \mathbb{N}_0$ and assume that $F_{n+1}(F_n + F_{n+2}) = \sum_{i=0}^n F_i^2 + F_{i+1}^2$. We add $F_{n+1}^2 + F_{n+2}^2$ to both sides. The RHS becomes $F_{n+1}^2 + F_{n+2}^2 + \sum_{i=0}^n F_i^2 + F_{i+1}^2 = \sum_{i=0}^{n+1} F_i^2 + F_{i+1}^2$. The LHS becomes $F_{n+1}(F_n + F_{n+2}) + F_{n+1}^2 + F_{n+2}^2 = F_{n+1}(F_n + F_{n+1}) + F_{n+2}(F_{n+1} + F_{n+2}) = F_{n+1}F_{n+2} + F_{n+2}F_{n+3} = F_{n+2}(F_{n+1} + F_{n+3})$. Putting it all together we get $F_{n+2}(F_{n+1} + F_{n+3}) = \sum_{i=0}^{n+1} F_i^2 + F_{i+1}^2$.

6. Let $a_n = 3n^2 + 7$. Prove or disprove that $a_n = \Theta(n^3)$.

   The statement is false, because $a_n \neq \Omega(n^3)$, which we now need to prove. Let $M \in \mathbb{R}, n_0 \in \mathbb{N}_0$. Choose $n = \max(10\lceil M \rceil + 1, n_0)$. This ensures that $n \in \mathbb{N}_0$, $n \geq n_0$, and also that $n > 10M \geq M|3 + \frac{7}{n^2}|$. Multiplying by (positive) n^2 we get $n^3 > M|3n^2 + 7|$, so $M|a_n| < |n^3|$.

7. Suppose that an algorithm has runtime specified by recurrence relation $T_n = 5T_{n/2} + n^2$. Determine what, if anything, the Master Theorem tells us.

We have $a = 5, b = 2$, and since $n^2 = \Theta(n^2)$, $k = 2$. We set $d = \log_b a = \log_2 5$, and note that $2 = \log_2 4 < \log_2 5 < \log_2 8 = 3$, so $2 < d < 3$. Since $k < d$, we are in the "small c_n" case of the Master Theorem, which tells us that $a_n = \Theta(n^d) = \Theta(n^{\log_2 5})$.

For problems 8-10, we consider a new function, "surround". For $x \in \mathbb{R}$, we define "the surround of x", writing $\boxed{x}$, as a natural number satisfying $\boxed{x}^2 - \boxed{x} \le |x| < \boxed{x}^2 + \boxed{x}$.

8. Prove uniqueness, i.e. $\forall x \in \mathbb{R}\ !\boxed{x} \in \mathbb{N},\ \ \boxed{x}^2 - \boxed{x} \le |x| < \boxed{x}^2 + \boxed{x}$.

Let $x \in \mathbb{R}$, and suppose $\boxed{x}, \boxed{y} \in \mathbb{N}$ with $\boxed{x}^2 - \boxed{x} \le |x| < \boxed{x}^2 + \boxed{x}$ and $\boxed{y}^2 - \boxed{y} \le |x| < \boxed{y}^2 + \boxed{y}$. We recombine to get $\boxed{x}^2 - \boxed{x} < \boxed{y}^2 + \boxed{y}$. Completing the square we get $(\boxed{x} - 0.5)^2 - 0.25 < (\boxed{y} + 0.5)^2 - 0.25$. Adding 0.25 and taking square roots (the positive square root since $\boxed{x}, \boxed{y} \ge 1$) we get $\boxed{x} - 0.5 < \boxed{y} + 0.5$, so $\boxed{x} < \boxed{y} + 1$. Applying Theorem 1.12(a) ("a theorem from the book") we get $\boxed{x} \le \boxed{y}$. We can recombine the other way to get $\boxed{y}^2 - \boxed{y} < \boxed{x}^2 + \boxed{x}$, and similar algebra gets us $\boxed{y} \le \boxed{x}$. Hence $\boxed{x} = \boxed{y}$.

NOTE: This problem is about proof structure, not algebra. The green portion of the solution was only worth 1 point. Here is an alternate version of the green portion, found by a student: Rewrite the inequality as $\boxed{x}^2 - \boxed{y}^2 < \boxed{x} + \boxed{y}$, factor as $(\boxed{x} + \boxed{y})(\boxed{x} - \boxed{y}) < \boxed{x} + \boxed{y}$. Now, since $\boxed{x} + \boxed{y} > 0$, we cancel to get $\boxed{x} - \boxed{y} < 1$. By Thm 1.12(a), $\boxed{x} - \boxed{y} \le 0$, so $\boxed{x} \le \boxed{y}$. We recombine the other way, and similar algebra gets us $\boxed{y} \le \boxed{x}$.

9. Prove existence, i.e. $\forall x \in \mathbb{R}\ \exists \boxed{x} \in \mathbb{N},\ \ \boxed{x}^2 - \boxed{x} \le |x| < \boxed{x}^2 + \boxed{x}$.

Let $x \in \mathbb{R}$. We use minimum element induction, defining $S = \{m \in \mathbb{Z} : m \ge 1 \wedge |x| < m^2 + m\} = \{m \in \mathbb{N} : |x| < m^2 + m\}$. It is a bit tricky to prove this is nonempty – we need to find some specific integer in S. One way is taking $m = \lceil |x| \rceil + 1$. Hence $|x| \le m < m + m^2$, and $m \ge 1$, so $m \in S$. S has a lower bound, namely 1. Induction gives us a minimum $n \in S$, so $|x| < n^2 + n$, and also $n - 1 \notin S$, so either $|x| \ge (n-1)^2 + (n-1) = n^2 - n$, or $n - 1 \not\ge 1$. If $n - 1 \not\ge 1$, then $n = 0$ and we also have $|x| \ge 0 = 0^2 - 0 = n^2 - n$. Combining, in both cases we get the desired double inequality $n^2 - n \le |x| < n^2 + n$.

10. Prove or disprove: $\forall x \in \mathbb{R}\ \forall k \in \mathbb{N},\ \ \boxed{x + k} = \boxed{x} + k$.

The statement is false, and requires a counterexample. Many are possible, here is just one. Take $x = 1$ and $k = 10$, we see that $\boxed{x} = 1$ since $1^2 - 1 \le |x| < 1^2 + 1$. We also have $\boxed{x + k} = 3$ since $3^2 - 3 \le |1 + 10| < 3^2 + 3$. However, $\boxed{x + k} = 3 \ne 11 = \boxed{x} + 10 = \boxed{x} + k$.